

The gravitational path integral and the trace of the diffeomorphisms

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I give a resolution of the conformal mode divergence in the Euclidean gravitational path-integral by isolating the trace of the diffeomorphisms and its contribution to the Faddeev-Popov measure.

I. INTRODUCTION

The ‘path-integral’ in quantum gravity is defined as a functional ‘integral’ over metrics. However, this ‘integral’ has not been completely obtained due to technical difficulties including the non-Gaussian nature of the integral as a function of the metric and the unboundedness of the Euclidean action [1] which appears in the ‘integrand’. There has been considerable research in-order to solve some of the problems, and in this paper we discuss the problem of the unboundedness of the Euclidean action. The Euclidean path integral is defined thus

$$\int \mathcal{D}g_{\mu\nu} \exp(-\Gamma_{\text{classical}}).$$

Where $g_{\mu\nu}$ represents a positive definite metric and $\Gamma_{\text{classical}}$ is Einstein’s action for gravity given by $\frac{-1}{16\pi G} \int \sqrt{\det g} R d^4x$. Here $\det g$ is the determinant of the metric $g_{\mu\nu}$, and R is the associated scalar curvature. The action can be written in terms of the conformal mode or the scale factor of the metric and a set of conformal equivalence class of metrics. The conformal mode ϕ of the metric can be isolated by writing $g_{\mu\nu} = e^{2\phi} \bar{g}_{\mu\nu}$ by using the Yamabe conjecture, where the conformal mode is fixed by requiring that the metric $\bar{g}_{\mu\nu}$ has a constant Ricci scalar. The action written in terms of these variables can be arbitrarily large and negative for a rapidly fluctuating conformal mode as the conformal mode gives a term $\frac{-3}{8\pi G} \int d^4x e^{2\phi} \sqrt{\bar{g}} (\bar{\nabla} \phi)^2$ to the Euclidean action ($\bar{\nabla}$ is a covariant derivative wrt the metric

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$\bar{g}_{\mu\nu}$). The term has the square of the derivative of the conformal mode, and thus the sign of the kinetic term is fixed for real ϕ . Metric configurations with unbounded derivatives of the conformal mode will contribute to make the Einstein action arbitrarily negative. Classically one can argue for positive action metrics, but in case of a quantum calculation, where the path-integral includes integration over arbitrary non-classical metrics, configurations with large negative actions would exist. The Euclidean path-integral which has the exponential of the negative of the Euclidean action is thus potentially divergent. Previous attempts to examine this particular problem [2–5], have concluded that the perturbative gravitational path integral when written in terms of the ‘physical variables’ has a positive definite effective action. This ‘identification’ of physical variables done by factoring out the diffeomorphism group from the measure is well defined in the perturbative regime or for metrics which are fluctuations over a classical metric. The identification of physical variables for the non-perturbative metrics remained a very difficult task.

In a path-integral, the physical metrics are identified in the measure in the integral by factoring out redundant configurations related by the diffeomorphism group. A set of metrics is identified as the physical metrics fixed by certain gauge conditions, and any other metric related to the physical set by diffeomorphisms is factored out of the measure. The physical measure now includes a Jacobian of the diffeomorphism transformations which relate the redundant metrics to the physical metrics. This Jacobian is known as the Faddeev-Popov determinant. The physical set of metrics are known as ‘gauge fixed metrics’ and the evaluation of the Faddeev-Popov determinant as a function of the metrics completes this procedure.

In [5] we used a formalism given in [4] to show that the Faddeev-Popov determinant for physical metrics defined using proper-time gauge constrains the metrics with large negative actions. The proper-time gauge was used as the path-integral could be defined using Lorentzian metrics and the analytically continued Euclidean metrics in the path-integral would have causal histories. The result was verified in the perturbative regime using an explicit calculation [5]. The non-perturbative regime was not calculated explicitly. In this paper, I define a new way of obtaining the physical measure and give an explicit non-perturbative calculation. This is achieved by writing the diffeomorphism transformations as comprising of a traceless part and a trace part. The conformal mode transforms due to the trace part of the diffeomorphisms, and I concentrate on the trace of the diffeomor-

phisms by parametrising the trace of the diffeomorphism by a scalar field. The terms in the measure which are due to factoring out the diffeomorphism generated by the scalar field are then isolated. Thus in some sense, I am using the Faddeev-Popov procedure only for the ‘scale’ or conformal sector of the metric. The Jacobian of the pure scale transformations is a scalar determinant which is then evaluated using heat kernel techniques for arbitrary non-perturbative metrics. This non-perturbative determinant makes the classical negative action positive. This result is true in any gauge and it appears that the resolution of the unboundedness of the Euclidean action arises from factoring out just the trace of the diffeomorphisms.

To summarise:

- 1) The gravitational action can assume arbitrarily negative values due to the kinetic term of the scale factor or the conformal mode of the metric.
- 2) The measure has redundant degrees of freedom due to the diffeomorphism group, and finding physical coordinates by factoring out the diffeomorphism group gives the Faddeev-Popov determinant. This comprises of scalar, vector and tensor determinants, and isolating the scalar determinant achieves the resolution of the conformal mode problem discussed in this paper.
- 3) We use the formalism of [6], where the physical metric is chosen by imposing gauge conditions on the metric $\bar{g}_{\mu\nu}$ and the Yamabe condition ensures that orbits of ϕ are transversal to the coordinates on the gauge slice. This approach is slightly different than the usual Faddeev-Popov ghost gauge fixing procedure where no such splitting of the metric is done. Thus the details of the gauge condition are encoded in the tensor Faddeev-Popov determinant which is due to gauge fixed $\bar{g}_{\mu\nu}$. The functional form of the scalar determinant (which originates from factoring out the trace of the diffeomorphisms) is same in any gauge, and in this paper this is obtained for arbitrary non-perturbative metrics using heat kernel techniques. Thus for the purposes of this paper, the details of the gauge are not relevant for the resolution of the unboundedness of the Euclidean action.

The actual calculation is executed in the following way: The path integral comprises of the measure $\mathcal{D}g_{\mu\nu}$ and the integrand $\exp(-\Gamma_{\text{classical}})$ where $\Gamma_{\text{classical}}$ is the classical gravitational action. We find the physical measure by dividing the given measure $\mathcal{D}g_{\mu\nu}$ by diffeomorphism group of the manifold \mathcal{M} , $Diff(\mathcal{M})$. The physical measure has a Faddeev-Popov

determinant, which is written in exponentiated form in the path-integral as $\exp(-\Gamma_1)$. Thus

$$\int \frac{\mathcal{D}g_{\mu\nu}}{\text{Diff}(\mathcal{M})} \exp(-\Gamma_{\text{classical}}) = \int \mathcal{D}g_{\mu\nu}^{\text{phys}} \exp(-\Gamma_1 - \Gamma_{\text{classical}}). \quad (1)$$

This is an integral over physical metrics with the weight corresponding to an effective action $\Gamma_{\text{effective}} = \Gamma_1 + \Gamma_{\text{classical}}$. The contribution from the physical measure Γ_1 can be split into that due to the trace of the diffeomorphisms Γ_{trace} , and the remaining ($\Gamma_{\text{traceless}}$). To find Γ_1 , I explicitly find a heat kernel of a non-Laplacian operator, which appears in the scalar Faddeev-Popov determinant. In the heat kernel calculation of the determinant, the regulator independent or ‘finite term’ is extracted, and is found to have the exact functional form as the classical action, but with a flipped sign in the regime where the curvature of space-time is slowly varying or $\nabla^\mu R_{\mu\nu} \sim 0$. In the regime where space-time is strongly curved or $\nabla^\mu R_{\mu\nu} \gg 1$ the conformal mode term is rendered positive but there appear higher order curvature terms (\bar{R}^2 and $\bar{\nabla}^\mu \bar{R}_{\mu\nu}$) contributing to the effective action. Thus the positive Γ_1 term when added to the negative $\sqrt{g}R$ term in the classical action renders the effective action positive definite. Thus this is not a counter term prescription where a regulator dependent badly divergent term is controlled by a counter term to get the physical parameters finite. The measure in the path-integral includes naturally a term which has the same functional form as the classical action with a opposite sign, and it reverses the overall sign of the $\int \sqrt{g}R$ term of the effective action. To clarify further this effective action is not obtained by integrating out matter fields coupled to gravity, but it is obtained by evaluating the determinants in the measure as a function of the physical metric.

In the next section the conformal mode problem in gravity is discussed, in section III the Faddeev-Popov measure is derived, in section IV, the resolution of the conformal mode divergence is obtained for the perturbative case as well as the non-perturbative case. The non-perturbative case includes the calculation of the heat kernel of the non-Laplacian operator, the details of which are derived in the Appendix.

II. THE CONFORMAL MODE IN GRAVITY

The Euclidean gravitational action or the action for positive definite metric comprises of

$$S = -\frac{1}{16\pi G} \int \sqrt{\det g} R d^4x \quad (2)$$

In [1] Hawking showed that if there is a particular decomposition of the metric of the form;

$$g_{\mu\nu} = e^{2\phi} \bar{g}_{\mu\nu} \quad (3)$$

where $e^{2\phi}$ is a conformal factor associated with the metric, and $\bar{g}_{\mu\nu}$ has a constant Ricci curvature \bar{R} , then the gravity action reduces to

$$S = -\frac{1}{16\pi G} \int d^4x e^{2\phi} \sqrt{\bar{g}} [\bar{R} + 6(\bar{\nabla}\phi)^2]. \quad (4)$$

The kinetic term of the conformal mode is positive definite, and hence the Euclidean action can assume as negative values as possible. This pathology can be assumed to be a signature of presence of redundant degrees of freedom, and indeed diffeomorphically related metrics in the measure are redundant. So, in this paper we investigate the path-integral written in terms of physical variables. To identify physical metrics, one takes a ‘gauge-slice’ in metric space, and other metrics related by pure diffeomorphism are factored out of the measure. This procedure, known as gauge fixing, leads to an effective action, re-written only in terms of the physical degrees of freedom. For perturbative gravity, it has been shown that [2], the physical action is indeed positive definite. For non-perturbative gravity, this gauge fixing is non-trivial, and it is difficult to find the ‘physical degrees of freedom’.

A clear procedure however exists for identifying the physical measure, using the Faddeev-Popov determinant. We use this and perform a non-perturbative calculation, to identify the ‘physical action’ for the trace sector of the theory. A generic metric can be written as

$$g_{\mu\nu} = e^{2\phi} \bar{g}_{\mu\nu} = e^{2\phi} \frac{\partial X^\lambda}{\partial x^\mu} \frac{\partial X^\rho}{\partial x^\nu} g_{\lambda\rho}^\perp \quad (5)$$

where $g_{\mu\nu}^\perp$ is a gauge fixed metric, and $X^\mu(x^\mu)$ is a diffeomorphism and ϕ is fixed by the Yamabe condition.

An infinitesimal version of this can be obtained in the cotangent space of the metric space as

$$h_{\mu\nu} = h_{\mu\nu}^\perp + (L\xi)_{\mu\nu} + \left(2\tilde{\phi} + \frac{1}{2}\nabla\xi\right) g_{\mu\nu} \quad (6)$$

Here $h_{\mu\nu}^\perp$ is the traceless part of the gauge invariant metric, ξ_μ the generator of diffeomorphisms, and the

$$(L\xi)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2}\nabla \cdot \xi g_{\mu\nu} \quad (7)$$

is an operator which maps vectors to traceless tensors. The last term of (6) corresponds to the trace sector of the metric including pure conformal orbits generated by $\tilde{\phi}$ representing an infinitesimal change in the conformal mode.

This coordinate transformation (6), when implemented in the measure in cotangent space leads to a Jacobian, the Faddeev-Popov determinant. The same determinant can be used for the measure in the path-integral, where a coordinate transformation to the physical coordinates $g_{\mu\nu}^\perp$ and conformal mode ϕ , and the diffeomorphisms ξ_μ is implemented [6]. In this paper we explicitly isolate the contribution of the trace part of the diffeomorphisms to the Faddeev-Popov determinant by parametrising the trace by a scalar field. This is described next.

A. Redefining Diffeomorphisms

The vector ξ which generates the diffeomorphism is broken up into a divergence less vector $\hat{\xi}$ and a divergence of a ‘scalar’ σ

$$\xi_\mu = \hat{\xi}_\mu + \nabla_\mu \sigma \quad (8)$$

$$\nabla^\mu \xi_\mu = \nabla^\mu \nabla_\mu \sigma \quad (9)$$

Thus the trace of the diffeomorphism $\nabla \cdot \xi$ is a function of the scalar σ . One more aspect of this discussion is that one can write $\hat{\xi}_\mu$ in terms of the divergence of an antisymmetric two tensor (this is discussed in [7])

The equation (9) determines σ up to a scalar whose divergence is zero, in terms of the trace of the diffeomorphism. The very interesting aspect of this breakdown is the fact that ‘orthogonal’ orbits of σ which do not contribute to the traceless part of the diffeomorphisms are not the ‘conformal killing’ orbits, but solutions to the equation

$$\left(\nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu\nu} \nabla^2 \right) \sigma = 0 \quad (10)$$

This does not reduce to the equation for solving for conformal killing vectors,

$$(L\xi)_{\mu\nu} = 0 \quad (11)$$

which is a much restricted equation for ξ_μ .

III. THE MEASURE IN THE PATH-INTEGRAL

We now calculate the Faddeev-Popov determinant, but using (8). The interesting aspect of this new calculation is the separation of the trace of the diffeomorphisms and evaluation

of the Faddeev-Popov determinant for this separately. The path-integral is defined to be

$$Z = \int \mathcal{D}\phi \mathcal{D}\bar{g}_{\mu\nu} \exp \left(\frac{1}{16\pi G} \int d^4x e^{2\phi} \sqrt{\bar{g}} [\bar{R} + 6(\bar{\nabla}\phi)^2] \right) \quad (12)$$

(we subsequently set $16\pi G=1$) To identify the correct measure one writes in the cotangent space of the De-Witt super space [6] or Equation (6), a coordinate transformation to the traceless gauge fixed part (denoted by a \perp) and a trace part and the pure diffeomorphisms generated by $\xi_\mu = \hat{\xi}_\mu + \nabla_\mu \sigma$. The Jacobian of the coordinate transformations is the Faddeev-Popov determinant and written as $(\det M)$ subsequently.

$$Z = \int \mathcal{D}\phi \mathcal{D}g_{\mu\nu}^\perp \mathcal{D}\hat{\xi}_\mu \mathcal{D}\sigma \det M \exp \left(\int d^4x \sqrt{\bar{g}} e^{2\phi} [\bar{R} + 6(\bar{\nabla}\phi)^2] \right) \quad (13)$$

This way of gauge fixing is completely non-perturbative, and is not specific to a gauge fixing. The $\det M$ is shown to have a scalar determinant (Appendix A) times a vector and a tensor determinant.

$$\det M = \det_S [8(1+2C) (-2(\nabla)^4 + 4\nabla_\mu \nabla^2 \nabla^\mu + 4\nabla_\mu \nabla_\rho \nabla^\mu \nabla^\rho)]^{1/2} \det_V \tilde{V} \det_T T \quad (14)$$

The operators V_μ and $T_{\mu\nu}$ which are vector and tensor operators can be easily identified as given in the appendix. ∇^μ is the covariant derivative operator and ∇^2 is the Laplacian with respect to the metric $g_{\mu\nu}$ (In case the derivative operators are evaluated for $\bar{g}_{\mu\nu}$ they are represented by $\bar{\nabla}$ operators). C is the constant in the De-Witt metric which determines the signature (see Appendix A for definition). The De-Witt metric is of indefinite signature for $C < -1/2$. The gauge volume or the integrations over the $\hat{\xi}_\mu$ and the σ can be taken out of the path-integral, and the Faddeev-Popov determinant remains to contribute to the effective action. Note that since this isolation of the trace is independent of the gauge fixing condition, our results will be true in any gauge.

IV. THE RESOLUTION OF THE CONFORMAL MODE PROBLEM

A. Perturbative Case

We begin by taking the perturbative case and give the resolution of the conformal mode problem in some known cases using the measure in (14). In the perturbative situation ϕ is taken to be very small, thus $e^{2\phi} = 1 + 2\phi + 2\phi^2$ and \bar{R} is the curvature of the space-time

one is perturbing about; hence the action gives

$$S = - \int d^4x \sqrt{g} (1 + 2\phi + 2\phi^2) [\bar{R} + 6(\bar{\nabla}\phi)^2] \quad (15)$$

$$= - \int d^4x \sqrt{g} [6(\bar{\nabla}\phi)^2 + 2\phi\bar{R} + 2\phi^2\bar{R} + \bar{R}] \quad (16)$$

$$= - \int d^4x \sqrt{g} \left\{ 2\phi'[-3\bar{\nabla}^2 + \bar{R}]\phi' + \bar{R} - \frac{1}{2}\bar{R}(\Delta^{-1})\bar{R} \right\} \quad (17)$$

Where the square in $\phi' = \phi + \frac{1}{2}\Delta^{-1}\bar{R}$ is completed and one obtains a non-local term in that process ($\Delta \equiv -3\bar{\nabla}^2 + \bar{R}$).

For perturbations about Minkowski space-time, $R_{\mu\nu\lambda\rho} = 0$ in the first approximation, and one obtains the scalar determinant (14) as

$$\det_S[8(1 + 2C)(6\bar{\nabla}^4)]^{1/2} \quad (18)$$

Under suitable boundary conditions, the determinant splits into

$$\det_S[-8(1 + 2C)6\bar{\nabla}^2] \det_S[-\bar{\nabla}^2] \quad (19)$$

The way we spilt the fourth order determinant, one corresponds to a convergent determinant $\det_s(-\bar{\nabla}^2)$, the other operator has divergent determinant for $C < -1/2$. The zero modes of the above determinants and the conformal killing directions, have to be factored to give a meaningful answer [4]. For the purposes of this paper, we assume that they have been factored as required. Thus the partition function acquires the following form in the conformal sector:

$$\int \mathcal{D}\phi \det_S[-48(1 + 2C)\bar{\nabla}^2]^{1/2} \det_S[-\bar{\nabla}^2]^{1/2} \exp(- \int d^4x \sqrt{g} \phi'[6\bar{\nabla}^2]\phi') \quad (20)$$

Clearly the ϕ integral can be written formally as a determinant, which is divergent.

$$\det_S[-48(1 + 2C)\bar{\nabla}^2]^{1/2} \det_S[-\bar{\nabla}^2]^{1/2} \frac{1}{\det_S[6\bar{\nabla}^2]^{1/2}} \quad (21)$$

Thus the divergent determinants from the Faddeev-Popov and the ϕ integral can be cancelled for $C < -1/2$. The remaining terms after the cancellation are convergent and will give finite answers for the path-integral.

Next the de-sitter and anti de sitter backgrounds which are constant curvature metrics and hence have $\bar{R}_{\mu\nu} = \Lambda g_{\mu\nu}$ are considered (Λ is a cosmological constant), and the conformal mode is treated as a fluctuation over that.

The scalar determinant (14) factorises rather neatly as fourth order operator is

$$6\nabla^4 + 4[\nabla_\mu, \nabla^2]\nabla^\mu + 4\nabla_\mu[\nabla_\rho, \nabla^\mu]\nabla^\rho = 6\nabla^4 + 8\Lambda\nabla^2 \quad (22)$$

The determinant can be factorised for the above operator using $\bar{R} = 4\Lambda$, as

$$\text{det}_S(-\bar{\nabla}^2)[8(1+2C)(-3\bar{\nabla}^2 - \bar{R})] = \text{det}_S(-\bar{\nabla}^2)\text{det}_S[8(1+2C)(-3\bar{\nabla}^2 - \bar{R})]. \quad (23)$$

The action for the scale factor in the de-sitter or anti-desitter backgrounds is

$$\int d^4x \sqrt{\bar{g}}(1+2\phi+2\phi^2) [\bar{R} + 6(\bar{\nabla}\phi)^2] + \int d^4x \sqrt{\bar{g}}(1+4\phi+8\phi^2)(-2\Lambda) \quad (24)$$

When the square is completed one obtains $\int d^4x \sqrt{\bar{g}} 2\phi'(-3\bar{\nabla}^2 - \bar{R})\phi'$ where we have put in $\bar{R} = 4\Lambda$. The integral over ϕ' thus gives a divergent determinant which is inverse to second of the factored determinants of (23) and is thus cancelled. Thus the partition function for pure conformal fluctuations about de Sitter space is given by

$$N[\text{det}_S(-\bar{\nabla}^2)]^{1/2} \quad (25)$$

Where N includes normalisation, but clearly the conformal mode divergence has been cancelled. So, for most Ricci flat and constant curvature metrics, we seem to have correctly identified the resolution. This resolution is similar to that described in [4]. There remain the transverse fluctuations of the metric, however which is beyond the scope of the discussion of the conformal mode.

B. Non-perturbative Case

For the non-perturbative case, no such neat factorisations of the determinant occur, and one has to compute the functional determinant using known techniques like the heat kernel equation. Ab initio, the evaluation of the determinant (14) in the non-perturbative regime appears very difficult, as the operator is a fourth order differential operator. The scalar operator (14) is clearly a self adjoint operator (the ∇^4 term is obviously self adjoint and the $\nabla_\mu R_{\mu\nu} \nabla_\nu$ is also self adjoint due to the symmetry of $R_{\mu\nu}$ under the interchange of the μ and the ν indices). But we can try to obtain the determinant using Heat Kernel techniques [8–10].

The heat kernel is defined in order to achieve a ζ function regularisation of the determinant. Given an operator F with eigenvalues λ , the zeta function is defined to be

$$\zeta(p) = \sum_{\lambda} \frac{1}{\lambda^p} \quad (26)$$

Clearly, the determinant of the operator, would be given by

$$\det(F) = e^{\text{Tr} \ln F} \quad (27)$$

and thus

$$\det(F) = e^{-\zeta'(0)}. \quad (28)$$

Other regularisation schemes can also be used if required, thus the exact answer would be particular to the way of regularisation. From (28), the Faddeev-Popov determinant can be written as an exponential and adds to the classical action in the ‘weight’ of the path-integral creating an effective action. Since the Faddeev-Popov has the square root of the determinant appearing in the measure, the exact terms which appear in the effective action for the scalar determinant is $\Gamma_{\text{trace}} = \frac{1}{2}\zeta'(0)$. Thus one has to find $\zeta'(0)$ for the scalar determinant of (14).

The zeta function for a given operator is appropriately written in terms of the ‘Heat Kernel’.

$$\zeta(p) = \frac{\mu^{2p}}{\Gamma(p)} \int dt t^{p-1} \text{Tr} \exp(-tF)(x, x') \quad (29)$$

The Heat Kernel is precisely the term $U(t, x, x') = \exp(-tF)(x, x')$, t is a parameter and x, x' represent coordinates of the manifold in which the operator is defined. The heat kernel also satisfies the differential equation

$$\left(\frac{\partial}{\partial t} + F \right) U(t, x, x') = \delta(x, x')|_{t=0} \quad (30)$$

where the boundary condition is that of a diffusion equation. The Heat Kernel can be solved exactly for the Laplacian in flat space, and for curved space-time, De-Witt wrote a particular ansatz, which has been solved partially. The ansatz for the Heat Kernel for the Laplacian is an expansion in powers of the parameter t

$$U(t, x, x') = \frac{1}{(4\pi t)^2} e^{-\frac{\bar{g}}{2t}} \sum_n a_n(x, x') t^n \quad (31)$$

where $\bar{\sigma}$ is half of the square of the geodesic distance existing given two points (x, x') . The actual evaluation of the heat kernel and the determinant of the operator is done in the coincidence limit $x \rightarrow x'$.

The procedure for finding the heat kernel for the scalar operator which appears in the determinant in the measure is simplified slightly, by solving in two different regimes. The scalar operator in (14) is taken thus

$$-2\nabla^4 + 4\nabla_\mu \nabla^2 \nabla^\mu + 4\nabla^\mu \nabla^\rho \nabla_\mu \nabla_\rho \quad (32)$$

By using the commutation relations, one obtains

$$6\nabla^4 + 4\nabla_\mu [\nabla^2, \nabla^\mu] + 4\nabla^\mu [\nabla^\rho, \nabla_\mu] \nabla_\rho \quad (33)$$

$$6\nabla^4 + 8\nabla_\mu R^{\mu\nu} \nabla_\nu \quad (34)$$

We take two limits, one where there is weak field gravity, and one obtains $\nabla_\mu R^{\mu\nu} \sim 0$, here (34) is approximated by

$$6\nabla^4 \left(1 + \frac{4}{3} (\nabla^4)^{-1} \nabla_\mu R^{\mu\nu} \nabla_\nu \right). \quad (35)$$

And where gravity is strong, one obtains

$$8\nabla_\mu R^{\mu\nu} \nabla_\nu \left(1 + \frac{3}{4} (\nabla_\mu R^{\mu\nu} \nabla_\nu)^{-1} \nabla^4 \right). \quad (36)$$

This particular approximation does not appear in any previous calculation, usually the expansion of higher order operators has been done in orders of $(\nabla^2)^{-1}$ [11]. Thus for the purposes of the kinetic term of the conformal mode, which is the reason for the unboundedness of the Euclidean action, it is enough to obtain the determinant of the following scalar operators in the above two limits.

$$\det_S \left(8(1 + 2C)6\nabla^4 \right) \quad (\nabla^\mu R_{\mu\nu} \sim 0) \quad (37)$$

and

$$\det_S (8(1 + 2C)8\nabla_\mu R^{\mu\nu} \nabla_\nu) \quad (\nabla^\mu R_{\mu\nu} \gg 1) \quad (38)$$

The first one (37) can be evaluated using the Heat Kernel for the Laplacian for arbitrary space times, under certain boundary conditions, by splitting the fourth order operator into product of two Laplacians whose heat kernel expansions are well known [8].

The (38) operator is clearly non-Laplacian type and non-minimal in the sense that $R_{\mu\nu}$ is not covariantly constant. Thus, I use a new ansatz for the Heat Kernel, where

$$U(t, x, x') = \frac{1}{(4\pi t)^2} \exp(-\varepsilon/2t) \sum_n a_n t^n \quad (39)$$

where ε , is a generalisation of $\bar{\sigma}$ used by De-Witt, and is determined by solution to an equation written below. One then finds a recursion relation for the a_i using the differential equation for the definition of the Heat Kernel (30).

The heat kernel differential equation for $t \neq 0$ gives for the operator $(-8\nabla^\mu R_{\mu\nu} \nabla^\nu)$

$$\left(\frac{\partial}{\partial t} - 8\nabla_\mu R^{\mu\nu} \nabla_\nu \right) \frac{1}{(4\pi t)^2} \exp(-\varepsilon/2t) \sum a_n t^n = 0 \quad (40)$$

$$\text{or } \left(\frac{\partial}{\partial t} - 8\nabla^\mu R^{\mu\nu} \nabla_\nu - 8R^{\mu\nu} \nabla_\mu \nabla_\nu \right) \frac{1}{(4\pi t)^2} \exp(-\varepsilon/2t) \sum a_n t^n = 0 \quad (41)$$

The linear term can be removed by a scaling by $\exp(\Lambda(x))$. The function Λ is determined according to the differential equation $\nabla_\tau(\Lambda) = -\frac{1}{2}(R^{-1})_{\nu\tau} \nabla_\lambda R^{\lambda\nu}$. This reduces the above equation to

$$\left(\frac{\partial}{\partial t} - 8R_{\mu\nu} \nabla^\mu \nabla^\nu + Q \right) \frac{1}{(4\pi t)^2} \exp(-\varepsilon/2t) \sum a_n t^n = 0 \quad (42)$$

where a potential $Q = -8\nabla_\mu R^{\mu\nu} \nabla_\nu \Lambda - 8R^{\mu\nu} \nabla_\mu \nabla_\nu \Lambda$ gets added due to the scaling of the wavefunction. The recursion relations for coefficients a_n obtained from (42) are:

$$\frac{\varepsilon}{2} - 2R_{\mu\nu} \nabla^\mu \varepsilon \nabla^\nu \varepsilon = 0 \quad (43)$$

$$8R_{\mu\nu} \nabla^\mu \nabla^\nu a_{n-1} - 8R_{\mu\nu} \nabla^\nu \varepsilon \nabla^\mu a_n - 4\nabla^\mu \nabla^\nu \varepsilon R_{\mu\nu} a_n - (n-2)a_n - Q a_{n-1} = 0 \quad (44)$$

The obvious solution for (43) is to take $\frac{1}{4}\varepsilon = R_{\mu\nu} \nabla^\mu \varepsilon \nabla^\nu \varepsilon$. This equation is solved easily in the case where the Ricci curvature is constant $R_{\mu\nu} = \frac{1}{l^2} g_{\mu\nu}$, and using that a generalisation is given for arbitrary curvature metrics in Riemann normal coordinates.

The case of the constant curvature (42) is solved by:

$$\frac{\varepsilon}{4} = \frac{1}{l^2} g_{\mu\nu} \nabla^\mu \varepsilon \nabla^\nu \varepsilon \quad (45)$$

$$\varepsilon = \frac{l^2}{8} \bar{\sigma} \quad (46)$$

where $\bar{\sigma}$ is half of the square of the geodesic distance from the point x to x' .

The case of arbitrary curvature, the Riemann normal coordinate expansion is taken for the metric and a solution is given in the coincidence limit only $x \rightarrow x'$. The Riemann normal

coordinates are used to expand for the metric and the curvature, about a given point, similar to a Taylor series for a function. The standard form for the square of the geodesic distances is

$$2\bar{\sigma}(x, x') = g_{\alpha\beta}\bar{\sigma}^\alpha\bar{\sigma}^\beta \quad (47)$$

where $\bar{\sigma}^\alpha(x, x')$ is the tangent vector to the geodesic at the point (x). In Riemann Normal coordinates, this is

$$2\bar{\sigma}(x, x') = \delta_{ab}\hat{x}^a\hat{x}^b \quad (48)$$

where \hat{x}^a is the coordinate joining the origin of the Riemann normal coordinates to a nearby point, or $(x - x')^a$ when x is taken as the origin and $x' \rightarrow x$. Motivated from equation (46), the ε can be written in Riemann normal coordinates as:

$$\varepsilon = \frac{1}{16}g_{ab}(R^{-1/2})^{ac}\hat{x}_c(R^{-1/2})^{bd}\hat{x}_d \quad (49)$$

This has the interesting properties in the coincidence limit (though there are no obvious interpretations for ε as there are for σ in terms of geodesics and tangents at the points x, x')

$$\nabla^a \varepsilon = 0 \quad (50)$$

$$\nabla^a \nabla^b \varepsilon = \frac{1}{8}(R^{-1})^{ab} \quad (51)$$

$$\nabla^a \nabla^b \nabla^c \varepsilon = 0 \quad (52)$$

$$\nabla^a \nabla^b \nabla^c \nabla^d \varepsilon = -\frac{1}{2}R^a{}_{e}{}^b{}_f(R^{-1/2})^{ec}(R^{-1/2})^{fd} \quad (53)$$

$(R^{-1/2})^{ac}$ is the square root of the inverse matrix of R_{ac} at the origin of the Riemann normal coordinates.

In the coincidence limit, one finds that, the recursion relation obtained in (44) reduces to

$$n a_n - 8R_{\mu\nu}\nabla^\mu\nabla^\nu a_{n-1} + Qa_{n-1} = 0 \quad (54)$$

To solve this recursion relation, we act on the split equation (44) again with the operator $R_{\mu\nu}\nabla^\mu\nabla^\nu$, and one gets a equation for $8R_{ab}\nabla^a\nabla^b a_n$ which is solved and substituted in (54). The coefficients a_o and a_1 are solved here.

As per the boundary condition, $a_0 = 1$. In addition, we get the the derivatives of a_0 from the coincidence limit of the equations in Appendix B.

$$\nabla^a a_0 = \frac{1}{2}(R^{-1})^{cd}\nabla^a R_{cd} \quad (55)$$

$$\begin{aligned}
R_{ab}\nabla^a\nabla^b a_0 &= -\frac{1}{4}R_{cd}[(R^{-1})^{ab}\nabla^c R_{ab}][(R^{-1})^{a'b'}\nabla^d R_{a'b'}] \\
&\quad - \frac{1}{2}R_{cd}(R^{-1})^{ab}\nabla^c\nabla^d R_{ab} - R^2
\end{aligned} \tag{56}$$

(In the coincidence limit the indices of the Riemann Normal coordinates can be interchanged with the μ indices). These equations are used in equation (54) to get a_1 . Similarly a a_n for arbitrary n can be obtained. The equation for a_1 is

$$a_1 = 8R_{\mu\nu}\nabla^\mu\nabla^\nu a_0 - Qa_0 \tag{57}$$

Substituting for Q and from (56), we get an expression for a_1 in terms of the curvature invariants and the derivatives of the curvature. This is thus a derivation of the Heat Kernel expansion of a non-Laplacian operator for non-perturbative gravity. In this derivation of the heat kernel expansion, we have ignored the boundary of the manifold, as we have done a Riemann Normal coordinate expansion about one local point. Inclusion of half-integer t coefficients ensures the inclusion of boundary terms, but that is not relevant for the calculation in this paper.

C. The non-perturbative resolution

In the two regimes we considered, $\nabla^\mu R_{\mu\nu} \sim 0$ and $\nabla^\mu R_{\mu\nu} \gg 1$, the terms in the Γ_{trace} or $\frac{1}{2}\zeta'(0)$ which will add to the $-\int d^4x e^{2\phi}\sqrt{\bar{g}}(\bar{\nabla}\phi)^2$ term in the classical action is the term with a_1 coefficient. The a_2 and higher coefficients are higher derivative terms and are not relevant for the discussion. So we will isolate the a_1 term in the zeta function's derivative, and see why we are convinced that the negative term in the classical action is taken care of by the measure's Γ_{trace} . The zeta function and its derivative are:

$$\zeta(p) = \frac{\mu^{2p}}{16\pi^2\Gamma(p)} \int dt t^{p-3} \text{Tr}_x e^{-tQ} \sum_i a_i t^i \tag{58}$$

$$\begin{aligned}
16\pi^2\zeta'(p) &= \frac{\mu^{2p}}{\Gamma(p)} \left(\ln \mu^2 - \frac{\Gamma'(p)}{\Gamma(p)} \right) \int dt t^{p-3} \text{Tr}_x e^{-tQ} \sum_i a_i t^i \\
&\quad + \frac{\mu^{2p}}{\Gamma(p)} \int dt t^{p-3} \ln t \text{Tr}_x e^{-tQ} \sum_i a_i t^i
\end{aligned} \tag{59}$$

The finite term as $p \rightarrow 0$ appears in the last term of the derivative of the zeta function (as we know this might not be the unique the way to extract the finite term but sure explains a way to cancel the negative term from the classical action). This is given in details in the

Appendix. The finite term (regulator independent) is remarkably proportional to a_1 and is obtained from (59) in the Appendix as (using a regularisation $\zeta(0) = -1/2$.)

$$\zeta'(0)_{\text{finite}} = -\frac{1}{32\pi^2} \text{Tr}_x a_1 \quad (60)$$

Thus in the effective action $\Gamma_{\text{trace}} + \Gamma_{\text{classical}} = \frac{1}{2}\zeta'(0) + \Gamma_{\text{classical}}$ we get to first order $-\frac{1}{64\pi^2} \text{Tr}_x a_1 + \Gamma_{\text{classical}}$. In the subsequent discussion, we fix the a_1 in the weak gravity and strong gravity regimes and find the $\Gamma_{\text{trace}} + \Gamma_{\text{classical}}$.

(i) $\nabla_\mu R_{\mu\nu} \approx 0$, the Faddeev-Popov determinant is

$$\det(8(1+2C)6\nabla^4) = \det(-8(1+2C)6\nabla^2)\det(-\nabla^2) \quad (61)$$

In the factorised form, the first scalar determinant is a divergent one for $C < -1/2$, and the second one is a convergent one. We discuss the divergent determinant's contribution to the effective action as this should cancel the divergence from the classical action.

The first scalar determinant is of a Laplacian and thus we use the heat kernel of a Laplacian, and analytically continue to the divergent regime of $C < -1/2$. The a_1 coefficient of the Laplacian is well known, and to quote [8, 9]

$$a_1 = \frac{1}{6}R \quad (62)$$

I scale the coefficient a_1 by Planck length squared ($Gh/2\pi$) to get the exponential dimensionless and restore the $16\pi G$ in the classical action. Writing R in terms of \bar{R} and $(\nabla\phi)^2$, and using the constants in the determinant one gets as the coefficient of the kinetic term of the conformal mode in the effective action ($\Gamma_{\text{trace}} + \Gamma_{\text{classical}}$)

$$-\frac{1}{16\pi} \left[1 + \frac{2(1+2C)}{\pi} \right] \quad (63)$$

Thus the positive action takes over at $(1+2C) > -\pi/2$. From Einstein' action $C = -2$, and Euclidean Einstein gravity has a positive definite effective action. The number $-\pi/2$ might differ for different regularisation schemes, (and conventions for defining determinants from Gaussian integrals) but it is indeed a finite number, and we should be in the realm of a convergent path-integral for Euclidean quantum gravity. Note that the effect of this calculation has been in the end change of the overall sign of the action by a minus sign. This is what has been observed in the continuum limit of the discrete lattice gravity calculation

of the path-integral by Ambjorn, Jurkiewicz and Loll [12]. So, these calculations confirm their observations of the effective action.

(ii) In the case of the regime $\nabla^\mu R_{\mu\nu} \gg 1$, the relevant operator is (38) and the coefficient a_1 is from (57, 56) (The Q in (57) is absorbed in the exponential of (59)),

$$\begin{aligned} a_1 = & -2R_{\lambda\sigma}[(R^{-1})^{\mu\nu}\nabla^\lambda R_{\mu\nu}][(R^{-1})^{\mu'\nu'}\nabla^\sigma R_{\mu'\nu'}] \\ & - 4R_{\lambda\sigma}(R^{-1})^{\mu\nu}\nabla^\sigma\nabla^\lambda R_{\mu\nu} - 8R^2 \end{aligned} \quad (64)$$

We find that $-R^2$ term has exactly the sign required to cancel the contribution from the $(\bar{\nabla}\phi)^2$ term in the classical action, as writing R in terms of \bar{R} and $\bar{\nabla}\phi$, one finds $(\bar{\nabla}\phi)^4$ from R^2 , which dominate for configurations with rapidly varying ϕ . The other terms do not give divergent negative terms. Thus $\Gamma_{\text{trace}} + \Gamma_{\text{classical}}$ in this regime also emerges as positive definite. Note in this non-perturbative regime, the effective action at this order does not merely change by an overall minus sign but has additional non-trivial contributions proportional to \bar{R}^2 and higher derivative terms.

V. DISCUSSIONS

In this paper, I isolated the scalar determinant in the Faddeev-Popov measure of the gravitational path integral and computed its contribution to the effective action using heat kernel techniques. It was found that this contribution added to the negative classical action to render it positive. In the process, I found the heat kernel of an operator $R_{\mu\nu}\nabla^\mu\nabla^\nu$, in the non-perturbative regime, and this is a useful result.

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Appendix A: Derivation of the Faddeev-Popov determinant

To derive the Faddeev-Popov determinant, for the particular gauge fixing of the gravitational path-integral, we use the method described in [4, 6]. The Gaussian normalisation condition fixes any ambiguity in the coefficients completely. We take the diffeomorphism transformations of the metric cotangent space elements and the new vector and scalar fields by

$$h_{\mu\nu} = h_{\mu\nu}^\perp + (L(\hat{\xi}, \sigma))_{\mu\nu} + (2\bar{\phi} + \frac{1}{2}\nabla^2\sigma)g_{\mu\nu} \quad (65)$$

Where

$$L(\hat{\xi}, \phi)_{\mu\nu} = \nabla_\nu \hat{\xi}_\mu + \nabla_\mu \hat{\xi}_\nu + \left(2\nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu\nu} \nabla^2 \right) \sigma \quad (66)$$

The Gaussian normalisation condition states that

$$1 = \int \mathcal{D}h_{\mu\nu} \exp \left(- \int \sqrt{g} d^4x h_{\mu\nu} G^{\mu\nu\rho\tau} h_{\rho\tau} \right) \quad (67)$$

where $G^{\mu\nu\rho\tau}$ is the DeWitt supermetric obtained in terms of the background metric as

$$G^{\mu\nu\rho\tau} = \frac{1}{2} (g^{\mu\rho} g^{\nu\tau} + g^{\mu\tau} g^{\nu\rho} + C g^{\mu\nu} g^{\rho\tau}) \quad (68)$$

Interchanging the coordinates of the tangent space leads to the determination of a Jacobian (a function of the metric), which is then determined as,

$$J^{-1} = \int \mathcal{D}h_{\mu\nu}^\perp \mathcal{D}\sigma \mathcal{D}\hat{\xi}_\mu \mathcal{D}\bar{\phi} \exp \left(- \int d^4x \sqrt{g} h_{\mu\nu} G^{\mu\nu\rho\tau} h_{\rho\tau} \right) \quad (69)$$

The scalar product in the exponent breaks up into

$$\begin{aligned} &= \int d^4x \sqrt{g} h_{\mu\nu}^\perp G^{\mu\nu\rho\tau} h_{\rho\tau}^\perp + \int \sqrt{g} d^4x L_{\mu\nu} G^{\mu\nu\rho\tau} L_{\rho\tau} + \int d^4x \sqrt{g} 2 h_{\mu\nu}^\perp G^{\mu\nu\rho\tau} L_{\rho\tau} \\ &+ 8(1 + 2C) \int d^4x \sqrt{g} \Omega^2 \end{aligned} \quad (70)$$

where $\Omega = \bar{\phi} + \frac{1}{4} \nabla^2 \sigma$, and can be just integrated as a redefinition of the conformal mode $\bar{\phi}$. Thus the Faddeev-Popov determinant gets only a factor of constant. The interesting terms are contained in

$$\begin{aligned} \int d^4x \sqrt{g} L_{\mu\nu} G^{\mu\nu\rho\tau} L_{\rho\tau} &= \int d^4x \sqrt{g} \left[\left\{ \nabla_\nu \hat{\xi}_\mu + \nabla_\mu \hat{\xi}_\nu + (2\nabla_\mu \nabla_\nu - \frac{1}{2} \nabla^2 g_{\mu\nu}) \sigma \right\} \right. \\ &\quad \times \left. G^{\mu\nu\rho\tau} \left\{ \nabla_\rho \hat{\xi}_\tau + \nabla_\tau \hat{\xi}_\rho + (2\nabla_\rho \nabla_\tau - \frac{1}{2} g_{\rho\tau} \nabla^2) \sigma \right\} \right] \end{aligned} \quad (71)$$

These terms give the Faddeev-Popov determinant from the vector and the scalar completion of squares. The completion of the square in the scalar sector gives this term:

$$\int d^4x \sqrt{g} \sigma' \left(2\nabla_\mu \nabla_\nu - \frac{1}{2} g_{\mu\nu} \nabla^2 \right) G^{\mu\nu\tau\sigma} \left(2\nabla_\tau \nabla_\sigma - \frac{1}{2} g_{\tau\sigma} \nabla^2 \right) \sigma' \quad (72)$$

The simplification of which gives

$$\int d^4x \sqrt{g} \sigma' \left[-2(\nabla)^4 + 4\nabla_\mu \nabla^2 \nabla^\mu + 4\nabla_\mu \nabla_\rho \nabla^\mu \nabla^\rho \right] \sigma' \quad (73)$$

where the $\sigma' = \sigma + X_\rho \hat{\xi}^\rho + Y^{\mu\nu} h_{\mu\nu}^\perp$ and $X_\rho = \Delta^{-1} [3\nabla_\rho \nabla^2 + R_\rho^\tau \nabla_\tau]$ and the operator $Y_{\mu\nu} = \frac{1}{2} \Delta^{-1} [2\nabla_\mu \nabla_\nu - \frac{1}{2} \nabla^2 g_{\mu\nu}]$. (Δ is the scalar operator which appears in the squared

term in (73)). The Integral is then completely Gaussian in each of the variables, and one obtains

$$J^{-1} = \frac{1}{\sqrt{8(1+2C)}} \int \mathcal{D}h_{\mu\nu}^{\perp} \mathcal{D}\sigma \mathcal{D}\hat{\xi}_{\mu} \exp \left(- \int d^4x \sqrt{g} \left[h_{\mu\nu}^{\perp} T^{\mu\nu\tau\rho} h_{\tau\rho}^{\perp} + \hat{\xi}_{\mu}' V^{\mu\nu} \hat{\xi}_{\nu}' + \sigma' \Delta \sigma' \right] \right) \quad (74)$$

The integration of the above gives the

$$J^{-1} = \frac{1}{\sqrt{8(1+2C)\det_{\text{T}}T\det_{\text{V}}V\det_{\text{S}}\Delta}} \quad (75)$$

Using the definition of determinants of operators. The operators T and V are tensor and vector operators, and hence their determinants are tensorial and vectorial determinants. We concentrate in obtaining the details of the scalar operator in this article.

Appendix B: The $R_{\mu\nu}\nabla^{\mu}\nabla^{\nu}$ operator

To find $\nabla^{\tau}a_0$ and $R_{\tau\sigma}\nabla^{\tau}\nabla^{\sigma}a_0$ which appear in the defining relation for a_1 , one takes the (54), and operates on it further with $R_{\mu\nu}\nabla^{\mu}\nabla^{\nu}$.

$$R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\sigma} \left[(n-2)a_n + 4\nabla^{\mu}\nabla^{\nu}\varepsilon R_{\mu\nu}a_n + 8\nabla^{\mu}\varepsilon\nabla^{\nu}a_n R_{\mu\nu} - 8R_{\mu\nu}\nabla^{\mu}\nabla^{\nu}a_{n-1} + Qa_{n-1} \right] = 0 \quad (76)$$

which results in

$$\begin{aligned} & (n-2)R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\sigma}a_n \quad (77) \\ & +4R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\sigma}\nabla^{\mu}\nabla^{\nu}\varepsilon R_{\mu\nu}a_n + 4R_{\lambda\sigma}\nabla^{\sigma}\nabla^{\mu}\nabla^{\nu}\varepsilon\nabla^{\lambda}R_{\mu\nu}a_n + 4R_{\lambda\sigma}\nabla^{\sigma}\nabla^{\mu}\nabla^{\nu}\varepsilon R_{\mu\nu}\nabla^{\lambda}a_n \\ & +4R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\mu}\nabla^{\nu}\varepsilon\nabla^{\sigma}R_{\mu\nu}a_n + 4R_{\lambda\sigma}\nabla^{\mu}\nabla^{\nu}\varepsilon\nabla^{\lambda}\varepsilon\nabla^{\sigma}R_{\mu\nu}a_n + 4R_{\lambda\sigma}\nabla^{\mu}\nabla^{\nu}\varepsilon\nabla^{\sigma}R_{\mu\nu}\nabla^{\lambda}a_n \\ & +4R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\mu}\nabla^{\nu}\varepsilon R_{\mu\nu}\nabla^{\sigma}a_n + 4R_{\lambda\sigma}\nabla^{\mu}\nabla^{\nu}\varepsilon\nabla^{\lambda}R_{\mu\nu}\nabla^{\sigma}a_n + 4R_{\lambda\sigma}\nabla^{\mu}\nabla^{\nu}\varepsilon R_{\mu\nu}\nabla^{\lambda}\nabla^{\sigma}a_n \\ & +8R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\sigma}\nabla^{\mu}\varepsilon\nabla^{\nu}a_n R_{\mu\nu} + 8R_{\lambda\sigma}\nabla^{\sigma}\nabla^{\mu}\varepsilon\nabla^{\lambda}\nabla^{\nu}a_n R_{\mu\nu} + 8R_{\lambda\sigma}\nabla^{\sigma}\nabla^{\mu}\varepsilon\nabla^{\nu}a_n\nabla^{\lambda}R_{\mu\nu} \\ & +8R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\mu}\varepsilon\nabla^{\sigma}\nabla^{\nu}a_n R_{\mu\nu} + 8R_{\lambda\sigma}\nabla^{\mu}\varepsilon\nabla^{\lambda}\nabla^{\nu}a_n\nabla^{\sigma}R_{\mu\nu} + 8R_{\lambda\sigma}\nabla^{\mu}\varepsilon\nabla^{\sigma}\nabla^{\nu}a_n\nabla^{\lambda}R_{\mu\nu} \\ & +8R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\mu}\varepsilon\nabla^{\nu}a_n\nabla^{\sigma}R_{\mu\nu} + 8R_{\lambda\sigma}\nabla^{\mu}\varepsilon\nabla^{\lambda}\nabla^{\nu}a_n\nabla^{\sigma}R_{\mu\nu} + 8R_{\lambda\sigma}\nabla^{\mu}\varepsilon\nabla^{\nu}a_n\nabla^{\lambda}\nabla^{\sigma}R_{\mu\nu} \\ & -8R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\sigma}R_{\mu\nu}\nabla^{\mu}\nabla^{\nu}a_{n-1} - 8R_{\lambda\sigma}\nabla^{\sigma}R_{\mu\nu}\nabla^{\lambda}\nabla^{\mu}\nabla^{\nu}a_{n-1} \\ & -8R_{\lambda\sigma}\nabla^{\lambda}R_{\mu\nu}\nabla^{\sigma}\nabla^{\mu}\nabla^{\nu}a_{n-1} - 8R_{\lambda\sigma}R_{\mu\nu}\nabla^{\lambda}\nabla^{\sigma}\nabla^{\mu}\nabla^{\nu}a_{n-1} + 8R_{\lambda\sigma}\nabla^{\lambda}\nabla^{\sigma}Qa_{n-1} \\ & +8R_{\lambda\sigma}\nabla^{\sigma}Q\nabla^{\lambda}a_{n-1} + 8R_{\lambda\sigma}\nabla^{\lambda}Q\nabla^{\sigma}a_{n-1} + 8R_{\lambda\sigma}Q\nabla^{\lambda}\nabla^{\sigma}a_{n-1} = 0 \end{aligned}$$

In the coincidence limit, apart from the terms containing $\nabla^\mu \nabla^\nu a_n$, there survives terms containing one derivative of coefficients, to determine one takes one derivative of the recursion relation

$$\begin{aligned} & (n-2)\nabla^\tau a_n + 4\nabla^\tau \nabla^\mu \nabla^\nu \varepsilon R_{\mu\nu} a_n + 4\nabla^\mu \nabla^\nu \varepsilon \nabla^\tau R_{\mu\nu} a_n \\ & + 4\nabla^\mu \nabla^\nu \varepsilon R_{\mu\nu} \nabla^\tau a_n + 8\nabla^\tau \nabla^\mu \varepsilon \nabla^\nu a_n R_{\mu\nu} + 8\nabla^\mu \varepsilon \nabla^\tau \nabla^\nu a_n R_{\mu\nu} \quad (78) \\ & + 8\nabla^\mu \varepsilon \nabla^\nu a_n \nabla^\tau R_{\mu\nu} - 8\nabla^\tau R_{\mu\nu} \nabla^\mu \nabla^\nu a_{n-1} - 8R_{\mu\nu} \nabla^\tau \nabla^\mu \nabla^\nu a_{n-1} + \nabla^\tau Q a_{n-1} + Q \nabla^\tau a_{n-1} = 0 \end{aligned}$$

In the coincidence limit, the non-zero terms from (77), are

$$\begin{aligned} & (n-2)R_{\mu\nu} \nabla^\mu \nabla^\nu a_n + 4R_{\lambda\sigma} \nabla^\lambda \nabla^\sigma \nabla^\mu \nabla^\nu \varepsilon R_{\mu\nu} a_n + 4R_{\lambda\sigma} \nabla^\mu \nabla^\nu \varepsilon \nabla^\lambda \nabla^\sigma R_{\mu\nu} a_n \quad (79) \\ & + 4R_{\lambda\sigma} \nabla^\mu \nabla^\nu \varepsilon \nabla^\sigma R_{\mu\nu} \nabla^\lambda a_n + 4R_{\lambda\sigma} \nabla^\mu \nabla^\nu \varepsilon \nabla^\lambda R_{\mu\nu} \nabla^\sigma a_n \\ & + 4R_{\lambda\sigma} \nabla^\mu \nabla^\nu \varepsilon R_{\mu\nu} \nabla^\sigma \nabla^\lambda a_n + 8R_{\lambda\sigma} \nabla^\sigma \nabla^\mu \varepsilon R_{\mu\nu} \nabla^\lambda \nabla^\nu a_n + 8R_{\lambda\sigma} \nabla^\lambda \nabla^\mu \varepsilon R_{\mu\nu} \nabla^\sigma \nabla^\nu a_n \\ & + 8R_{\lambda\sigma} \nabla^\lambda \nabla^\nu \varepsilon \nabla^\sigma R_{\mu\nu} \nabla^\nu a_n - 8R_{\lambda\sigma} \nabla^\lambda \nabla^\sigma R_{\mu\nu} \nabla^\mu \nabla^\nu a_{n-1} - 8R_{\lambda\sigma} \nabla^\sigma R_{\mu\nu} \nabla^\lambda \nabla^\mu \nabla^\nu a_{n-1} \\ & + 8R_{\lambda\sigma} \nabla^\lambda \nabla^\sigma Q a_{n-1} + 8R_{\lambda\sigma} \nabla^\lambda Q \nabla^\sigma a_{n-1} + 8R_{\lambda\sigma} \nabla^\lambda Q \nabla^\sigma a_{n-1} + 8R_{\lambda\sigma} Q \nabla^\lambda \nabla^\sigma a_{n-1} = 0 \end{aligned}$$

Putting equations (50,51,52,53) in the above one gets (55,56).

Appendix C: The zeta function derivative

As given in equation (59), the derivative of the zeta function is

$$\begin{aligned} 16\pi^2 \zeta'(p) &= \frac{\mu^{2p}}{\Gamma(p)} \left[\ln \mu^2 - \frac{\Gamma'(p)}{\Gamma(p)} \right] \int dt \, t^{p-3} \, \text{Tr}_x \, e^{-Qt} \sum_i a_i t^i \\ &+ \frac{\mu^{2p}}{\Gamma(p)} \int dt \, t^{p-3} \, \ln t \, \text{Tr}_x e^{-Qt} \sum_i a_i t^i \quad (80) \end{aligned}$$

The last term gives the integral to be

$$\begin{aligned} &= \frac{\mu^{2p}}{\Gamma(p)} \int dt \, t^{p-3} \, \ln t \, \text{Tr}_x e^{-Qt} \sum_i a_i t^i \\ &= \frac{\mu^{2p}}{\Gamma(p)} \int dt \, t^{p-3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (t-1)^n \text{Tr}_x e^{-Qt} \sum_i a_i t^i \\ &= \frac{\mu^{2p}}{\Gamma(p)} \int dt \, t^{p-3} \sum_{n=1}^{\infty} \sum_{j=0}^n \frac{(-1)^{n+1}}{n} {}^n C_j t^j (-1)^{n-j} \text{Tr}_x e^{-Qt} \sum_i a_i t^i \\ &= \frac{\mu^{2p}}{\Gamma(p)} \sum_{i,j,n} \frac{(-1)^{2n+1-j}}{n} \Gamma(p-2+i+j) {}^n C_j \text{Tr}_x a_i Q^{2-p-i-j} \quad (81) \end{aligned}$$

The term which is finite and non-zero in the above is obtained for $i = 1$ and $j = 1$, and plugging this in the derivative, and taking the limit $p \rightarrow 0$ is precisely

$$\sum_{n=1}^{\infty} (1) \text{Tr}_x a_1 = \zeta(0) \text{Tr}_x a_1 \quad (82)$$

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